arXiv:physics/0003063 23 Mar 2000

Energy Balance in an Electrostatic Accelerator

Max S. Zolotorev
Center for Beam Physics, Lawrence Berkeley National Laboratory, Berkeley, CA 94720
Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(Feb. 1, 1998)

1 Problem

The principle of an electrostatic accelerator is that when a charge e escapes from a conducting plane that supports a uniform electric field of strength E_0, then the charge gains energy eE_0d as it moves distance d from the plane. Where does this energy come from?

Show that the mechanical energy gain of the electron is balanced by the decrease in the electrostatic field energy of the system.

2 Solution

Once the charge has reached distance d from the plane, the static electric field $\mathbf{E}_e$ at an arbitrary point $\mathbf{r}$ due to the charge can be calculated by summing the field of the charge plus its image charge,

$$\mathbf{E}_e(\mathbf{r}, d) = \frac{e\mathbf{r}_1}{r_1^3} - \frac{e\mathbf{r}_2}{r_2^3}, \tag{1}$$

where $\mathbf{r}_1$ ($\mathbf{r}_2$) points from the charge (image) to the observation point $\mathbf{r}$, as illustrated in Fig. 1. The total electric field is then $E_0\hat{\mathbf{z}} + \mathbf{E}_e$.

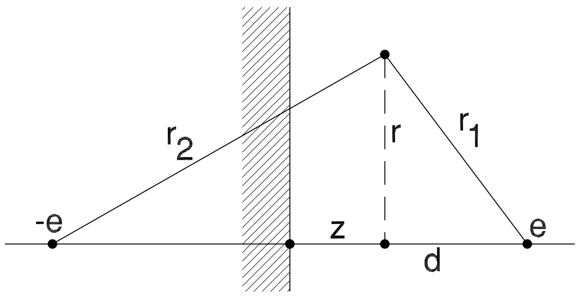

Figure 1: The charge e and its image charge $-e$ at positions $(r, \theta, z) = (0, 0, \pm d)$ with respect to a conducting plane at $z = 0$. Vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are directed from the charges to the observation point $(r, 0, z)$.

It turns out to be convenient to use a cylindrical coordinate system, where the observation point is $\mathbf{r} = (r, \theta, z) = (r, 0, z)$, and the charge is at $(0, 0, d)$. Then,

$$r_{1,2}^2 = r^2 + (z \mp d)^2. \tag{2}$$

The part of the electrostatic field energy that varies with the position of the charge is the interaction term (in Gaussian units),

$$
\begin{aligned}
U_{\rm int} &= \int \frac{E_0 \hat{\mathbf{z}} \cdot \mathbf{E}_{\rm e}}{4\pi} d{\rm Vol} \\
&= \frac{eE_0}{4\pi} \int_0^\infty dz \int_0^\infty \pi dr^2 \left(\frac{z-d}{[r^2+(z-d)^2]^{3/2}} - \frac{z+d}{[r^2+(z+d)^2]^{3/2}} \right) \\
&= \frac{eE_0}{4} \int_0^\infty dz \left(\left\{ \begin{array}{ll} 2 & \text{if } z>d \\ -2 & \text{if } z<d \end{array} \right\} - 2 \right) \\
&= -eE_0 \int_0^d dz = -eE_0 d. \qquad (3)
\end{aligned}
$$

When the particle has traversed a potential difference $V = E_0 d$, it has gained energy eV and the electromagnetic field has lost the same energy.

In a practical "electrostatic" accelerator, the particle is freed from an electrode at potential $-V$ and emerges with energy eV in a region of zero potential. However, the particle could not be moved to the negative electrode from a region of zero potential by purely electrostatic forces unless the particle lost energy eV in the process, leading to zero overall energy change. An "electrostatic" accelerator must have an essential component (such as a battery) that provides a nonelectrostatic force that can absorb the energy extracted from the electrostatic field while moving the charge from potential zero, so as to put the charge at rest at potential $-V$ prior to acceleration.